ON THE ISOMORPHISM BETWEEN THE DUALIZING SHEAF AND THE CANONICAL SHEAF

MATTHEW H. BAKER AND JÁNOS A. CSIRIK

Abstract. We give a new proof of the isomorphism between the dualizing sheaf and the canonical sheaf of a non-singular projective variety X over a perfect field k. Our proof uses concepts and results from algebraic number theory.

1. Introduction

Recall the definition of a dualizing sheaf from [3, III, 7]. If X is a proper scheme of dimension n over a field k, a *dualizing sheaf* for X is a coherent sheaf ω on X, together with a trace morphism $t : H^n(X, \omega) \to k$ such that for all coherent sheaves $\mathcal{F}$ on X, the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \omega) \times H^n(X, \mathcal{F}) \to H^n(X, \omega)$$

followed by t gives an *isomorphism*

$$\mathrm{Hom}(\mathcal{F}, \omega) \xrightarrow{\sim} H^n(X, \mathcal{F})'.$$

If a dualizing sheaf for X exists, then it is unique up to canonical isomporphism by [3, III, 7.2].

An explicit calculation shows that $\mathbb{P}^n$ has a dualizing sheaf $\omega_{\mathbb{P}^n}$. If X is any projective variety over a perfect field k, then a dualizing sheaf ω_X exists and can be constructed as follows. By Noether normalization, there is a finite separable morphism $f : X \to \mathbb{P}^n$. Then $\omega_X = f^! \omega_{\mathbb{P}^n}$ is a dualizing sheaf for X, where $f^!$ is a functor whose definition is given in Section 2. This construction readily leads to a proof of the Serre duality theorem [3, III, 7.6], different from the one given in [3, III, §7].

In Section 3, which is the main part of this paper, we give a direct proof that the dualizing sheaf ω_X constructed above is isomorphic to the canonical sheaf if X is nonsingular. The proof uses a number of facts about Dedekind domains and discrete valuation rings. In particular, the theory of the different plays an important role. The reader familiar with algebraic number theory will find the present approach an intriguing alternative to the homological methods used in [3, III, §7]. Yet another proof of our main result, Corollary 9, can be found in [4, 5].

All references in round brackets will be to [3].

1991 *Mathematics Subject Classification.* 14F10.

Key words and phrases. Serre Duality, Dualizing Sheaf, Differentials.

The authors would like to thank H. W. Lenstra, Jr. for his valuable assistance. Special debt is owed to R. Hartshorne for suggesting this project to us.

The first author was supported by an NDSEG fellowship.

2. Serre duality

Let $f : X \to Y$ be a finite morphism of noetherian schemes. For any quasi-coherent sheaf $\mathcal{G}$ on Y, the sheaf $\mathcal{H}\mathrm{om}_Y(f_*\mathcal{O}_X, \mathcal{G})$ is a quasi-coherent $f_*\mathcal{O}_X$-module, so it corresponds to a quasi-coherent $\mathcal{O}_X$-module that we shall call $f^!\mathcal{G}$ (see (III, Ex. 6.10(a))). Before we go on, let us consider the case where $f : X = \operatorname{Spec} B \to Y = \operatorname{Spec} A$ corresponds to $f^\# : A \to B$. For some terminology, if A and B are any commutative rings, with $N \in A\text{-}\mathfrak{Mod}$ and $M \in (A, B)\text{-}\mathfrak{BiMod}$, we define $\underline{\mathrm{Hom}}_{\mathrm{A}}(M, N)$ to be the usual Hom but with the obvious B-module structure derived from M. Then the functor $f^!$ corresponds to $\underline{\mathrm{Hom}}_{\mathrm{A}}(B, -)$.

Returning to the general (not necessarily affine) case, the functor $f^!$ behaves like a right adjoint to f_* in the sense that for any $\mathcal{F} \in \mathfrak{Coh}(X)$ and $\mathcal{G} \in \mathfrak{Qco}(Y)$, there is a natural isomorphism

$$f_*\mathcal{H}\mathrm{om}_X(\mathcal{F}, f^!\mathcal{G}) \to \mathcal{H}\mathrm{om}_Y(f_*\mathcal{F}, \mathcal{G}) \tag{1}$$

(see (III, Ex. 6.10(b))). Taking global sections in (1), we obtain the isomorphism

$$\mathrm{Hom}_X(\mathcal{F}, f^!\mathcal{G}) \cong \mathrm{Hom}_Y(f_*\mathcal{F}, \mathcal{G}). \tag{2}$$

For future reference, note that a simple calculation using (II, Ex. 5.2) gives the following lemma:

Lemma 1. *Let $f : X \to Y$ be a finite morphism of noetherian schemes, and $\mathcal{M}$ a locally free sheaf on Y. Then*

$$f^!\mathcal{M} \cong f^!\mathcal{O}_Y \otimes_{\mathcal{O}_X} f^*\mathcal{M}. \tag{3}$$

We now sketch how the Serre duality theorem (III, 7.6) can be proved for an integral projective scheme X of dimension n over a perfect field k using the functor $f^!$. Let us first recall the following lemma.

Lemma 2 (Noether Normalization)**.** *Let X be an integral projective scheme of dimension n over a perfect field k. Then there exists a finite separable morphism $f : X \to \mathbb{P}^n$.*

Proof. By (I, 4.8A), $K(X)/k$ is separably generated, so by (I, 4.7A) it contains a separating transcendence base, yielding an injection $K(\mathbb{P}^n) = k(Y_1, \ldots, Y_n) \hookrightarrow K(X)$. By (I, 4.4), this gives a dominant morphism $f : X \to \mathbb{P}^n$, which is finite and separable, since $K(X)/K(\mathbb{P}^n)$ is. ∎

Now let X be as above. By Lemma 2, there exists a finite separable morphism $f : X \to \mathbb{P}^n$. We define $\omega_X := f^!\omega_{\mathbb{P}^n}$, where $\omega_{\mathbb{P}^n}$ is the dualizing sheaf for $\mathbb{P}^n$ (see (III, 7.1)). By (III, Ex. 7.2), ω_X is a dualizing sheaf for X. (Note that by (III, 7.2), any two dualizing sheaves for X are canonically isomorphic, so ω_X does not depend on the choice of f.) Since f is flat if and only if X is Cohen–Macaulay by [2, Exer. 18.17 (or Cor. 18.17)], the Serre duality theorem (III, 7.6) now follows from (III, Ex. 6.10).

3. The dualizing sheaf

Now we will show that the dualizing sheaf $\omega_X := f^!\omega_{\mathbb{P}^n}$ is in fact the canonical sheaf on a nonsingular (noetherian) projective scheme X. Recall from (III, 7.1) that for $\mathbb{P}^n$, the canonical sheaf $\bigwedge^n \Omega_{\mathbb{P}^n}$ is the dualizing sheaf. Before stating our next lemma, we need to recall the following definition.

Definition 3. *If $\mathcal{T}$ is a torsion sheaf on a normal noetherian scheme X, we define the* ramification divisor *of $\mathcal{T}$ to be*

$$R = \sum_Z \operatorname{length}_{\mathcal{O}_\zeta}(\mathcal{T}_\zeta) \cdot Z,$$

where the sum ranges over all the irreducible closed subschemes Z of codimension one in X, and ζ denotes the generic point of Z.

Lemma 4. *Let X be a normal noetherian scheme. Suppose $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves of rank n on X and that $\mathcal{T}$ is a torsion sheaf on X with ramification divisor R, such that the sequence*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{T} \longrightarrow 0$$

is exact. Then $\bigwedge^n \mathcal{G} \cong \bigwedge^n \mathcal{F} \otimes \mathcal{L}(R)$.

Proof. We make the following definition: if M is a module of finite length over some commutative ring A, then we denote by $\chi_A(M)$ the product of the (not necessarily distinct) primes occuring in a Jordan-Hölder series for M. (For noetherian integrally closed domains, this χ_A is the canonical map from the category of finite length A-modules to its Grothendieck group, which is the group of ideals of A. See [7, I §5] for details.) Also note that since X is normal, it makes sense to talk about Weil divisors on X.

We now begin the proof of the lemma. Taking nth exterior powers, we get an exact sequence

$$0 \longrightarrow \textstyle\bigwedge^n \mathcal{F} \xrightarrow{\det} \textstyle\bigwedge^n \mathcal{G} \longrightarrow \mathcal{C} \longrightarrow 0.$$

Since $\bigwedge^n \mathcal{G}$ is locally free of rank 1, we can tensor by its dual to get an exact sequence

$$0 \longrightarrow \textstyle\bigwedge^n \mathcal{F} \otimes (\bigwedge^n \mathcal{G})^{-1} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{C} \otimes (\bigwedge^n \mathcal{G})^{-1} \longrightarrow 0.$$

But then $\bigwedge^n \mathcal{F} \otimes (\bigwedge^n \mathcal{G})^{-1}$ equals (as subsheaves of $\mathcal{O}_X$) $\mathcal{I}_D$, the ideal sheaf of some locally principal closed subscheme of X corresponding to an (effective) Cartier divisor D, and hence $\mathcal{C} \cong (\bigwedge^n \mathcal{G}) \otimes \mathcal{O}_D$.

If we let $C(X)$ be the group of Cartier divisors on X, then there is an injection $C(X) \hookrightarrow \operatorname{Div}(X)$, where $\operatorname{Div}(X)$ is the group of Weil divisors. We claim that $R = D$ as Weil divisors. It suffices to show that $n_Z = n'_Z$ for each prime divisor Z of X, where n_Z denotes the coefficient of Z in R, and n'_Z denotes the coefficient of Z in D. Fix one such Z, and denote by ζ its generic point. Then the local ring $\mathcal{O}_\zeta$ is a discrete valuation ring, and we have the following exact sequences of $\mathcal{O}_\zeta$-modules:

$$0 \longrightarrow \mathcal{F}_\zeta \xrightarrow{u} \mathcal{G}_\zeta \longrightarrow \mathcal{T}_\zeta \longrightarrow 0$$

and

$$0 \longrightarrow \textstyle\bigwedge^n \mathcal{F}_\zeta \xrightarrow{\det u} \textstyle\bigwedge^n \mathcal{G}_\zeta \longrightarrow \mathcal{C}_\zeta \longrightarrow 0.$$

By the theory of modules over a principal ideal domain, we have

$$n_Z = \operatorname{length}_{\mathcal{O}_\zeta} \mathcal{T}_\zeta = \operatorname{length}_{\mathcal{O}_\zeta}(\operatorname{coker} u) = v_Z\left(\chi_{\mathcal{O}_\zeta}(\operatorname{coker} u)\right),$$

where v_Z denotes the valuation on the local ring $\mathcal{O}_\zeta$. By [7, I, §5, Lemma 3], we have $\chi_{\mathcal{O}_\zeta}(\operatorname{coker} u) = \beta \cdot \mathcal{O}_\zeta$, where $\beta \cdot \mathcal{O}_\zeta$ is the image of $\det u$. So $n_Z = v_Z(\beta) = \operatorname{length}_{\mathcal{O}_\zeta}(\mathcal{O}_D)_\zeta = n'_Z$ as claimed. This shows that $D = R$ as Weil divisors, and hence as Cartier divisors; in particular, R is a well-defined Cartier divisor! We conclude that

$$\mathcal{L}(-R) \cong \mathcal{I}_D \cong \bigwedge^n \mathcal{F} \otimes \left(\bigwedge^n \mathcal{G} \right)^{-1},$$

so $\bigwedge^n \mathcal{G} \cong \bigwedge^n \mathcal{F} \otimes \mathcal{L}(R)$. ■

Remark: Using the terminology of (II, Ex. 6.11(b)), we have proved that the determinant $\det \mathcal{T}$ of our sheaf $\mathcal{T}$ is isomorphic to $\mathcal{L}(R)$.

We now want to prove the following:

Theorem 5. *Let $f : X \to Y$ be a finite separable morphism between integral nonsingular schemes of dimension n. Then*

$$\Omega_X^n \cong f^! \Omega_Y^n.$$

PROOF. What we will end up doing is proving in a slightly roundabout way that

$$\Omega_X^n \cong f^* \Omega_Y^n \otimes \mathcal{L}(R) \tag{4}$$

and

$$f^! \Omega_Y^n \cong f^* \Omega_Y^n \otimes \mathcal{L}(R) \tag{5}$$

for the same "ramification divisor" R.

Let us first try to prove (4). By (II, 8.11), we have an exact sequence

$$f^* \Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0.$$

Now Ω_X is locally free of rank n on X, and Ω_Y is locally free of rank n on Y, so $f^*\Omega_Y$ is locally free of rank n on X (since $f^*\mathcal{O}_Y = \mathcal{O}_X$). As f is a finite separable morphism, $K(X)/K(Y)$ is a finite separable extension so by [6, Thm. 25.3], we get $\Omega_{K(X)/K(Y)} = 0$. Hence $\Omega_{X/Y}$ is a torsion sheaf. Now letting $\mathcal{K}$ be the kernel of $f^*\Omega_Y \to \Omega_X$, we get an exact sequence at each point P of X

$$0 \to \mathcal{K}_P \to \mathcal{O}_P^n \to \mathcal{O}_P^n \to (\Omega_{X/Y})_P \to 0.$$

Tensoring this sequence with $K(X)$, which is flat over $\mathcal{O}_P$, we get

$$\dim_{K(X)} \mathcal{K}_P \otimes K(X) = -n + n + \dim_{K(X)} (\Omega_{X/Y})_P \otimes K(X) = 0,$$

so $\mathcal{K}_P$ is torsion. But $\mathcal{K}_P \subseteq \mathcal{O}_P^n$, and $\mathcal{O}_P$ is an integral domain, so we have $\mathcal{K} = 0$, i.e.

$$0 \to f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0.$$

Now taking the ramification divisor R associated to $\Omega_{X/Y}$, we have

$$\Omega_X^n \cong f^* \Omega_Y^n \otimes \mathcal{L}(R)$$

by Lemma 4, proving (4). Assertion (5) will be proved as Lemma 8. ■

In order to prove Lemma 8, some further preparation is needed. Let $f : X \to Y$ again be a finite separable morphism between nonsingular irreducible schemes X and Y. Then X is in particular Cohen-Macaulay, so $f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module (and also is an $\mathcal{O}_Y$-algebra via $f^\#$).

We want to define an $\mathcal{O}_Y$-linear "trace" map $\mathrm{Tr} : f_*\mathcal{O}_X \to \mathcal{O}_Y$ as follows. On any sufficiently small affine open $\mathcal{U} = \operatorname{Spec} A \in Y$, we have $f^{-1}\mathcal{U} = \operatorname{Spec} B$ affine, and B is a free A-module of rank $d := \deg f$. Let $\{e_1, \ldots, e_d\}$ be a basis for B over A, and for $b \in B$ write $be_i = \sum a_{ij} e_j$. Then we can set $\mathrm{Tr}(b) := \sum a_{ii}$ on B, which is independent of the chosen basis, so the local maps defined in this way glue properly to give a global map of sheaves on X. (For further discussion see [1, Chapter VI, Remark 6.5].)

By (2), the map $\mathrm{Tr} \in \mathrm{Hom}_Y(f_*\mathcal{O}_X, \mathcal{O}_Y)$ gives rise to a map $\overline{\mathrm{Tr}} \in \mathrm{Hom}_X(\mathcal{O}_X, f^!\mathcal{O}_Y)$. We will be interested in the cokernel $\mathcal{F}$ of the morphism $\overline{\mathrm{Tr}}$. We first collect some facts about Dedekind rings and then prove a proposition characterizing the sheaf $\mathcal{F}$.

Lemma 6. *Let A and B be Dedekind domains with fraction fields K and L, respectively. Suppose that L is finite separable over K, with B the integral closure of A in L. Then*

(a) $B \otimes_A K = L$

(b) $\underline{\mathrm{Hom}}_A(B, A) \hookrightarrow \underline{\mathrm{Hom}}_K(L, K)$

(c) If I, J are fractional ideals of B, $I/IJ \cong B/J$.

(d) If M is a torsion B-module, then M may be written uniquely in the form $M = B/I_1 \oplus \cdots \oplus B/I_m$, where $I_1 \subseteq \cdots \subseteq I_m$ is an ascending sequence of nontrivial ideals of B.

Proof. (a) Both L and $B \otimes_A K$ can be characterized as the integral closure of K in L; (b) follows from (a); (c) is an exercise for the reader; (d) see [2, Ex. 19.6(c)]. ■

Proposition 7. *$\overline{\mathrm{Tr}}$ is injective, and its cokernel $\mathcal{F}$ is a torsion sheaf. Moreover, the ramification divisor of $\mathcal{F}$ is equal to the ramification divisor of $\Omega_{X/Y}$.*

Proof. We begin with a sequence of reductions. Since the statement of the lemma is of local nature and f is a finite morphism, we can assume that $Y = \mathrm{Spec}\, A$ and $X = \mathrm{Spec}\, B$. In fact, it suffices to prove the proposition after localizing at a height 1 prime of A; we may thus assume that $(A, \mathfrak{m})$ is a discrete valuation ring and B is a semilocal ring with maximal ideals $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ dominating $\mathfrak{m}$. As f is surjective (and hence dominant), $f^\# : A \to B$ is injective, so we may assume without loss of generality that $A \subseteq B$.

By the equivalence of categories between B-modules and quasicoherent sheaves on $\mathrm{Spec}\, B$, we find that the exact sequence

$$\mathcal{O}_X \xrightarrow{\overline{\mathrm{Tr}}} f^!\mathcal{O}_Y \longrightarrow \mathcal{F} \longrightarrow 0 \tag{6}$$

corresponds to an exact sequence of B-modules

$$B \xrightarrow{\overline{\mathrm{Tr}}} B^* \longrightarrow M \longrightarrow 0, \tag{7}$$

where $B^* = \underline{\mathrm{Hom}}_{\mathrm{A}}(B, A)$.

The image of $1 \in B$ under $\overline{\mathrm{Tr}}$ is easily seen to be $\mathrm{Tr} \in B^*$. If we let K and L denote the fields of fractions of A and B respectively, then L/K is separable (since f is), so $\mathrm{Tr} \neq 0 \in B^*$ by [7, III, §3]. This shows that $\overline{\mathrm{Tr}}$ is injective.

Let K and L be the fraction fields of A and B, respectively. Note that $K = K(Y)$ and $L = K(X)$. By assumption A is Dedekind, and since B is finite over A it is integral over A. But X is nonsingular, hence normal, so B is integrally closed and thus is the integral closure of A in L. By (I, 6.3A) B is also a Dedekind ring, so applying Lemma 6 it follows that $B \otimes_A K = L$, and that we have an injection

$B^* = \underline{\mathrm{Hom}}_{\mathrm{A}}(B, A) \hookrightarrow \underline{\mathrm{Hom}}_K(L, K)$. But $\underline{\mathrm{Hom}}_K(L, K)$ can be canonically identified with L via

$$\begin{array}{ccc} L & \xrightarrow{\sim} & \underline{\mathrm{Hom}}_K(L, K) \\ x & \mapsto & (l \mapsto \mathrm{Tr}_{L/K}(xl)). \end{array}$$

Under this identification, B^* corresponds to

$$B^\dagger = \{x \in L : \mathrm{Tr}_{L/K}(xB) \subseteq A\} \subseteq L.$$

Here $B^\dagger$ is a fractional ideal of B containing B; its inverse, the *different* $\mathfrak{D}_{B/A}$, is an integral ideal of B [7, III, §3]. By Lemma 6(c), we have

$$B^\dagger / B \cong B / \mathfrak{D}_{B/A}.$$

Note also that since B is a Dedekind ring, B^* is locally free of rank 1 as a B-module. Hence $f^! \mathcal{O}_Y$ is a locally free sheaf of rank 1, and we in fact see that $\mathcal{F}$ is a torsion sheaf in (6):

$$0 \longrightarrow \mathcal{O}_X \longrightarrow f^! \mathcal{O}_Y \longrightarrow \mathcal{F} \longrightarrow 0. \tag{8}$$

We now claim that $\mathrm{length}_{\mathcal{O}_\zeta} (\Omega_{X/Y})_\zeta = \mathrm{length}_{\mathcal{O}_\zeta} \mathcal{F}_\zeta$ for all ζ corresponding to generic points of prime divisors Z on X. So we have to show that

$$\mathrm{length}_{B_\mathfrak{p}} M_\mathfrak{p} = \mathrm{length}_{B_\mathfrak{p}} (\Omega_{B/A})_\mathfrak{p} \tag{9}$$

for height 1 primes $\mathfrak{p} \subseteq B$, i.e. $\mathfrak{p}_1, \ldots \mathfrak{p}_r$. But this follows from [8, Theorem 4.1(1)]. (Note that since L/K is separable, $\Omega_{B/A} \otimes_B L = \Omega_{L/K} = 0$, hence $\Omega_{B/A}$ is a torsion B-module.) ■

Remark: By Lemma 6(d), we must have $\Omega_{B/A} = \bigoplus_1^t B/I_i$ for some ideals I_i in B. Then we can see from [8] that $\mathfrak{D}_{B/A}$ can be expressed as $\mathfrak{D}_{B/A} = \prod_1^t I_i$.

Remark: Note that if we knew that all $k(\zeta)/k(\eta)$ were *separable* (where $\eta \in Y$ corresponds to $\mathfrak{m} \in \mathrm{Spec}\, A$), then we could apply [7, III, §7 Prop. 14] to obtain this result.

We have all the components in place now for the proof of (5):

Lemma 8. *Let R be the ramification divisor of $\mathcal{F}$ (which, by Proposition 7, is the same as the ramification divisor of $\Omega_{X/Y}$). Then the following isomorphism holds.*

$$f^! \Omega_Y^n \cong f^* \Omega_Y^n \otimes \mathcal{L}(R).$$

Proof. Using the short exact sequence (8), we obtain, by Lemma 4,

$$f^! \mathcal{O}_Y \cong \mathcal{L}(R).$$

Tensoring this isomorphism on both sides by the sheaf $f^* \Omega_Y^n$, we see that

$$f^! \mathcal{O}_Y \otimes_{\mathcal{O}_X} f^* \Omega_Y^n \cong f^* \Omega_Y^n \otimes \mathcal{L}(R).$$

But since Ω_Y^n is locally free, by Lemma 1 we get

$$f^! \Omega_Y^n \cong f^* \Omega_Y^n \otimes \mathcal{L}(R).$$

■

This establishes (5), so the proof of Theorem 5 is now complete.

Corollary 9. *Let X be a nonsingular projective variety of dimension n over k. Then $\omega_X \cong \Omega_X^n$.*

PROOF. Let $f : X \to \mathbb{P}^n$ be a finite separable morphism whose existence is guaranteed by Lemma 2. By definition, $\omega_X \cong f^!\omega_{\mathbb{P}^n}$. But $\omega_{\mathbb{P}^n} \cong \Omega^n_{\mathbb{P}^n}$. The result now follows from Theorem 5. ∎

References

[1] Allen Altman and Steven Kleiman, *Introduction to Grothendieck Duality Theory*, Lecture Notes in Mathematics 146, Springer-Verlag, 1970
[2] David Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics 150, Springer-Verlag, 1995
[3] Robin Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer-Verlag, 1977
[4] Ernst Kunz, *Holomorphe Differentialformen auf algebraischen Varietäten mit Singularitäten I,* Manuscripta math., **15** (1975), 91–108
[5] Ernst Kunz, *Differentialformen auf algebraischen Varietäten mit Singularitäten II,* Abh. Math. Sem. Univ. Hamburg, **47** (1978), 42–70
[6] Hideyuki Matsumura, *Commutative Ring Theory*, Cambridge studies in advanced mathematics 8, Cambridge University Press, 1986
[7] Jean-Pierre Serre, *Local Fields*, Graduate Texts in Mathematics 67, Springer-Verlag, 1979
[8] Bart de Smit, *The different and differentials of local fields with imperfect residue fields*, Proc. Edinburgh Math. Soc. (2), **40** (1997), no. 2, 353–365

E-mail address: `baker@math.berkeley.edu`

E-mail address: `janos@math.berkeley.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720-3840